NUMBER THEORY AND DIOPHANTINE ANALYSIS.

157. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for m and n in $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$.

No solution of this problem has been received.

158. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Find positive rational values of a and b in the equation $x^4-2ax^2+x+a^2-b=0$, that will make each of the roots (all different) rational numbers.

Solution by E. B. ESCOTT, Ann Arbor, Mich.

Let two roots be $x=\alpha$ and $x=\beta$. Substituting, we have

$$b=\alpha^4-2a\,\alpha^2+\alpha+a^2=\beta^4-2a\,\beta^2+\beta+a^2.$$

Transposing, and removing factor $\alpha-\beta$ (since $\alpha\neq\beta$), we have $\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3-2a(\alpha+\beta)+1=0$, *i. e.*,

$$a=\tfrac{1}{2}\left(\alpha^2+\beta^2+\frac{1}{\alpha+\beta}\right).$$

Substituting, we get $b=\frac{1}{4}\left(\alpha^2-\beta^2-\frac{1}{\alpha+\beta}\right)^2+\alpha$.

Substituting in the original equation, and removing the factors $x-\alpha$ and $x-\beta$, we have

$$x^2+(\alpha+\beta)x+\left(\alpha\beta-\frac{1}{\alpha+\beta}\right)=0.$$

This will have commensurable roots if

$$(\alpha+\beta)^2-4\left(\alpha\beta-\frac{1}{\alpha+\beta}\right)=(\alpha-\beta)^2+\frac{4}{\alpha+\beta}=r^2.$$

It is easily seen that α and β cannot be integral.

We can get as many rational values as we please by assuming any value for $\alpha+\beta$.

Example. Let $\alpha+\beta=\frac{1}{2}$. Then $\beta=\frac{1}{2}-\alpha$. Substituting in the last equation, we get $(2\alpha-\frac{1}{2})^2+8=r^2$. This can be satisfied in an infinite number of ways, *e. g.*,

$$2\alpha-\tfrac{1}{2}+r=4,\quad 2\alpha-\tfrac{1}{2}-r=-2,$$

whence $\alpha=\frac{3}{4}$, $\beta=-\frac{1}{4}$. Then $a=\frac{21}{16}$, $b=\frac{21}{16}$. The other roots are $\frac{5}{4}$ and $-\frac{7}{4}$.

Also solved by G. B. M. Zerr, and V. M. Spunar.

AVERAGE AND PROBABILITY.

192a. Proposed by A. H. HOLMES, Brunswick, Maine.

In a game of baccarat the dealer and each side of the table have two or three cards. The object is to get as near nine as possible, and tens and court cards do not count. If the first two cards dealt do not together amount to five, the player asks for another. If above five he does not. When the two cards amount to exactly five would the chances of the hand be bettered or diminished by drawing a third card, and how much?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let A, B, C, D be the players, and, in order to avoid a multiplicity of solutions, we will assume cards for A, C, D when B has just five. Let A have an ace and a three; C, four and six; D, two and five, as follows:

A	C	D	B
1, 3	4, 6	2, 5	1, 4 or 2, 3 or 5, 10

In either case B betters his hand if he draws 1, 2, 3, 4, 5, 6, or 7; diminishes it if he draws 9 and leaves it the same if he draws 8, 10, or court cards. Since there are but 44 cards left in the pack, we have:

First $\begin{cases} \frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{4}{44}=\frac{10}{22}=\frac{5}{11}=\text{chance of bettering.} \\ \frac{4}{44}=\frac{1}{11}=\text{chance of diminishing.} \\ \frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}=\frac{5}{11}=\text{chance of leaving same.} \end{cases}$

Second $\begin{cases} \frac{3}{44}+\frac{2}{44}+\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{4}{44}=\frac{5}{11}=\text{chance of bettering.} \\ \text{The chance of diminishing or leaving the same} \\ \text{is } \frac{1}{11} \text{ or } \frac{5}{11} \text{ as before.} \end{cases}$

Third $\begin{cases} \frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{2}{44}+\frac{3}{44}+\frac{4}{44}=\frac{21}{44}=\text{chance of bettering.} \\ \text{The chance of diminishing}=\frac{4}{44}=\frac{1}{11}. \\ \frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{3}{44}=\frac{19}{44}=\text{chance of remaining the same.} \end{cases}$

The solution is similar for A, C, D having other cards.

Hence we see that the drawing of a third card when the two cards count just five, is preferable. In the first and second cases, bettering to diminishing$=5:1$. In the third case, bettering to diminishing$=21:4$.

197. Proposed by HENRY HEATON, Belfield, N. D.

Solve No. 188 on the supposition that all lines having the same direction are equally distributed in space, and lines passing through the same point are distributed as the radii of a sphere drawn to points equally distributed.